INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

SECOND CLASS CONSTRAINTS AND THEIR ELIMINATION IN THE QUANTUM THEORY

A.Y. Shiekh

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1988 MIRAMARE-TRIESTE

IC/88/52

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SECOND CLASS CONSTRAINTS AND THEIR ELIMINATION
IN THE QUANTUM THEORY *

A.Y. Shiekh
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

It is shown how second class constraints are inconsistent, even classically, and methods for their elimination are discussed in a quantum mechanical context.

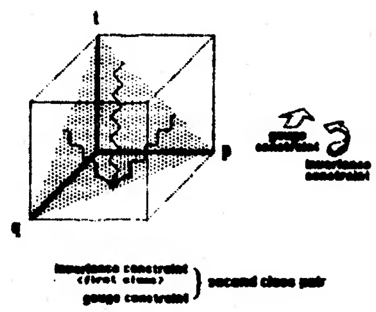
MIRAMARE - TRIESTE
March 1988

* To be submitted for publication.

Dealing with Second Class Constraints in the Quantum Theory

Dirac's viewed second class constraints as inconsistent, and he proposed a scheme to eliminate them via now so called Dirac brackets [Dirac, 1964]. Batalin and Fradkin [1986], however, take up a differing outlook. To appreciate this a small digression is called for:

Note that first class constraints are generators of gauge transformations [Dirac, 1964] and become second class under gauge fixing[1]. This is illustrated below:

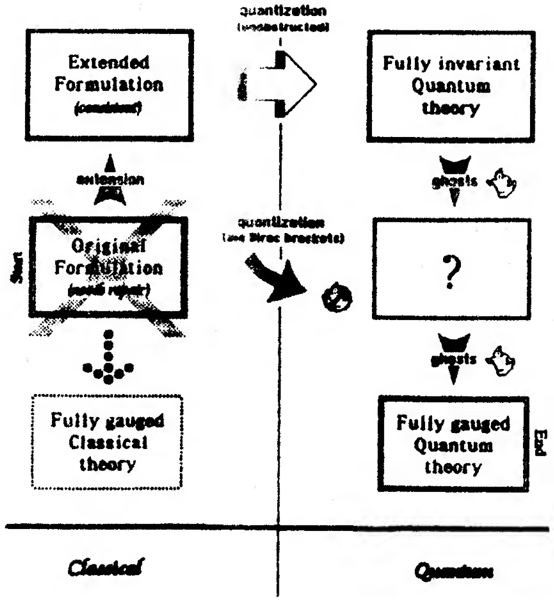


Constrained Phase Space

This leads to the view that the presence of second class constraints might be seen as indicating some degree of gauge fixing, and that one should then search for the fully ungauged theory (having no second class constraints) which when partially gauge fixed reduces to the original.

This route, in general, is significantly different from the Dirac bracket scheme since all the gauging is now to be done in the quantum regime where Faddeev Popov gauge fixing factors are picked up. In general such factors do not decouple and serve the purpose of implementing correct gauge fixing. This idea is illustrated below:

[1] The gauge fixing introduces further constraints that remove the first class (commuting) behaviour (first class constraints being the generators of gauge symmetry). This is the reason, also, why second class constraints come in even numbers; since an invariance gives rise to one constraint and its gauge fixing another.



Quantization of a Second Class System

An example of interest is the Green-Schwarz superstring action [Green, Schwartz and Witten, 1987], which although manifestly covariant, is plagued by second class constraints. Under the Dirac bracket procedure, manifest covariance is lost and this motivates the search for a fully first class, manifestly covariant, rendition.

Generalized Hamiltonian Formulation

A constrained system is characterized by a Hamiltonian of the form [Dirac, 1964]:

$$H = H_0 + \sum_n \lambda_n \phi_n \qquad H_0 = \sum_n p_n \dot{q}_n - L$$

where an **arbitrary** variation of the q_n, p_n and λ_n leads to the equations of motion:

$$\dot{q}_n = \frac{\partial H}{\partial p_n} \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n}$$

and the so called primary constraints:

$$\phi_n \approx 0$$

where $\approx$ means equality to by virtue of the constraints (weakly equal).

However, this is not the end of the story since there are consistency conditions to be satisfied. For the constraints to be maintained for all times:

$$\dot{\phi}_n = (\phi_n, H) \approx 0$$

i.e.

$$(\phi_n, H_0) + \lambda_m(\phi_n, \phi_m) \approx 0$$

This might imply further (so called secondary) constraints[1]. If all the constraints are first class (commute[2] with the total Hamiltonian for all λ), then the Lagrange multipliers are truly arbitrary (as they must be) and the system is in working condition. However, if the constraints are second class (do not commute), then conditions are imposed upon the Lagrange multipliers in contradiction to their arbitraryness, and the Hamiltonian as it stands is therefore inconsistent. Dirac showed a way to eliminate these second class constraints via the now so called 'Dirac bracket' [Dirac, 1964], which yields a first class (and so consistent) Hamiltonian formulation of the system. The alternative is to apply all the conditions on the Lagrange multipliers which then also leads to a first class formulation, albeit different. The restrictions upon the Lagrange multipliers fall into two distinct classes:

I) $\lambda \approx 0$ This is a constrained system in the most usual sense [Goldstein, 1980; Landau and Lifshitz, 1976], with a force (λ) maintaining the constrained motion

[1] to be attached to the Hamiltonian with associated Lagrange multipliers and which must also undergo the consistency conditions

[2] Here 'commute' is used in the sense of zero Poisson bracket

II) $\lambda = 0$ This is a condition selectively **removing** the second class constraints

The Dirac constraint procedure has recognized second class constraints as gauge fixing conditions and has removed them to yield an extended first class system. This is of no virtue if the gauging is to be done classically, since one then still has to deal with second class constraints. But the system is now first class and so quantizable, be it via path integral or operator quantization. If the gauge fixing is then performed in the quantum regime [Faddeev and Popov, 1967; Fradkin and Vilkovisky, 1975], the troubles associated with second class constraints are not encountered. An alternative approach for dealing with the gauge group volume without having to make a gauge choice has recently been proposed [Shiekh, 1987]. A discussion of consistent quantization has also been dealt with [Shiekh, 1987].

Having set out the scheme, a better grasp might be obtained by looking at a very simple example.

Prototype Second Class System

Much understanding can be gained by investigating an example stripped of superfluous detail. So analysing the classical system characterized by the Lagrangian:

$$L = q_2 \dot{q}_1$$

The equations of motion then follow as:

$$\dot{q}_2 = 0 \qquad \dot{q}_1 = 0$$

Now move to the Hamiltonian formalism:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = q_2 \qquad p_2 = \frac{\partial L}{\partial \dot{q}_2} = 0$$

which therefore has the primary constraints:

$$\chi_1 = p_1 - q_2 = 0$$
$$\chi_2 = p_2 = 0$$

These are second class (non commuting) constraints, since $(\chi_1, \chi_2) \neq 0$, by virtue of $(q_m, p_n) = \delta_{mn}$, $(q_m, q_n) = 0$, $(p_m, p_n) = 0$, where the Poisson bracket is defined by:

$$(f,g) = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}$$

So the naive Hamiltonian is obtained as:

$$H_0 = p_1 \dot{q}_1 + p_2 \dot{q}_2 - L$$
$$= 0$$

which leads to the total Hamiltonian:

$$H = \lambda_1 (p_1 - q_2) + \lambda_2 (p_2)$$

where the λ_m are Lagrange multipliers.

The Hamilton equations of motion follow as:

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0 \qquad \dot{q}_1 = \frac{\partial H}{\partial p_1} = \lambda_1$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = \lambda_1 \qquad \dot{q}_2 = \frac{\partial H}{\partial p_2} = \lambda_2$$

with the constraints:

$$\chi_1 = p_1 - q_2 = 0 \qquad \chi_2 = p_2 = 0$$

which are in agreement with the Lagrange equations of motion, as they must be.

However, this is not the end of the story since it remains to check consistency of the constraints with the equations of motion, to give:

$$\dot{\chi}_1 = (\chi_1,H) = -\lambda_2 = 0 \quad \text{and} \quad \dot{\chi}_2 = (\chi_2,H) = \lambda_1 = 0$$

In general, if the constraints do not commute among themselves (are second class), then they cannot commute with the Hamiltonian, and so consistency then places conditions on the Lagrange multipliers. Application of these conditions leads, in this case, to a null Hamiltonian, which does not regenerate the parent Lagrangian, and so it would seem that another way must be found to deal with the second class constraints.

A hint of the flaw, and its possible correction, is given from the constraints:

$$q_2 = p_1 \quad \text{and} \quad p_2 = 0$$

which suggests that q_2, rather than being an independent coordinate, is actually the conjugate momentum to q_1.

Not unexpectedly, since second class constraints are not consistent in the classical Hamiltonian formulation, they are also seen as an obstruction to quantization.

In the path integral the obstruction takes the form of not allowing the Hamiltonian to be written in the form:

$$H_0 + \lambda_m \phi_m$$

(λ_m arbitrary) since this implies [Dirac, 1964] that H_0 and the ϕ_m are first class[1].

In the operator formalism the quantization prescription[2] $(\, , \,) \rightarrow \frac{1}{\hbar}[\, , \,]$ leads to a contradiction since second class constraints, by definition, do not commute.

$$[\hat{\chi}_m, \hat{\chi}_n] \neq \hat{0}$$

but

$$\hat{\chi}_m \approx \hat{0} \qquad \hat{\chi}_n \approx \hat{0}$$

A general technique for eliminating second class constraints was proposed by Dirac [1964]. An alternative and more recent approach, proposed by Batalin and Fradkin [1987], will also be considered.

[1] second class formulations having non arbitrary Lagrange multipliers as a result of the consistency conditions

[2] which is itself ambiguous [Chernoff, 1981]

Reducing the System

(The Dirac Bracket)

The Dirac bracket is defined by [Dirac, 1964]:

$$(\varepsilon,\eta)_{DB} = (\varepsilon,\eta) - (\varepsilon,\chi_\alpha)\,(M^{-1})_{\alpha\beta}\,(\chi_\beta,\eta)$$

where:

$$M = \begin{bmatrix} 0 & (\chi_1,\chi_2) & \cdots & (\chi_1,\chi_m) \\ (\chi_2,\chi_1) & 0 & \cdots & \\ \vdots & \vdots & & \vdots \\ (\chi_m,\chi_1) & & \cdots & 0 \end{bmatrix}$$

and χ_m are the second class constraints which are then eliminated by substitution, to yield a first class Hamiltonian formulation of the system.

Applying this to the original example:

$$M = \begin{bmatrix} 0 & (p_1-q_2,p_2) \\ (p_2,p_1-q_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Hence:

$$(q_1,q_2)_{DB} = 1$$

which confirms the former result that q_2 is the conjugate momentum to q_1, i.e. that one is already in the Hamiltonian formalism.

Check this explicitly ($q = q_1$, $p = q_2$):

$$H = p\dot{q} - L$$
$$= 0$$

Equations of motion:

$$\dot{p} = -\frac{\partial H}{\partial q} = 0 \qquad (\dot{q}_2 = 0)$$

$$\dot{q} = \frac{\partial H}{\partial q} = 0 \qquad (\dot{q}_1 = 0)$$

which is that to be demonstrated.

Extending the System

With all this understanding in mind one might turn back to the original Lagrange formulation that seemed flawed in Hamiltonian form; for consistency led to a null Hamiltonian:

$$H = 0$$

which although leading to a first class system and reproducing the correct equations of motion:

$$\dot{p}_1 = 0 \qquad \dot{q}_1 = 0$$
$$\dot{p}_2 = 0 \qquad \dot{q}_2 = 0$$

would seem not to regenerate the original Lagrangian:

$$\begin{aligned} L &= p_1\dot{q}_1 + p_2\dot{q}_2 - H \\ &= 0 \end{aligned}$$

But the removal of second class constraints indicates the Hamiltonian possesses extended gauge freedom. This is confirmed by the ability to pick the originally inconsistent constraints as the gauge:

$$p_1 = q_2 \qquad p_2 = 0$$

which then leads back to the original Lagrangian:

$$L = q_2\dot{q}_1$$

To avoid returning to a second class Hamiltonian formulation, the second class constraints are to be applied as gauge fixing conditions after quantization where such troubles do not appear.

It is intriguing just how intelligent the Dirac constraint procedure is in fixing up a second class system. Being compelled to deliver a first class system it extends the gauge freedom necessary to turn second class constraints to first.

Summary

When second class constraints (which indicate an inconsistency) occur in the transition to the Hamiltonian formalism one may eliminate them via Dirac brackets, or apply the conditions they imply upon the Lagrange multipliers. The latter leads to a first class extended system which requires extra gauging. The gauging may then be performed after quantizing[1] (which is now unobstructed) with the associated 'ghost' degrees of freedom [Faddeev and Popov, 1967; Fradkin and Vilkovisky, 1975] which would be missed if gauging classically with the help of Dirac brackets.

Acknowledgements

This work grew from a talk given by Professor E.Fradkin while visiting the International Centre for Theoretical Physics. The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

[1] so avoiding the generation of second class constraints

References

1. I.Batalin and E.Fradkin, 'Operatorial Quantisation of Dynamical Systems subject to Second class Constraints', Nucl.Phys. **B279**, (1987) 514-, and references therein.

2. P.Chernoff, 'Mathematical Obstructions to Quantisation', Had.J. **4**, (1981), 879-.

3. P.Dirac, '**Lectures on Quantum mechanics**', Belfer Graduate School of Science, 1964, and references therein.

4. E.Fradkin and G.Vilkovisky, 'Quantisation of Relativistic Systems with Constraints', Phys.Lett. **55B**, (1975) 224-

   see also

   '**Quantum Field Theory and Quantum Statistics, Essays in Honour of the Sixtieth Birthday of E.S.Fradkin**' Eds I.Batalin, C.Isham and G.Vilkovisky, Adam Hilger, 1987

5. H.Goldstein '**Classical Mechanics**', 2nd ed., Addison-Wesley, 1980

6. M.Green, J.Schwarz and E.Witten, '**Superstring Theory**', Cambridge University Press, 1987, and references therein.

7. L.Faddeev and V.Popov, 'Feynman diagrams for the Yang-Mills field', Phys.Lett. **25B**, (1967) 29-

8. R.Feynman, 'Space-Time Approach to Quantum Mechanics', Rev.Mod.Phys. **20** (1948) 267- (reprinted in '**Quantum Electrodynamics**', Ed J.Schwinger, Dover, 1958)

9. R.Feynman and A.Hibbs, '**Quantum Mechanics and Path Integrals**', McGraw-Hill, 1965

10. L.Landau and E.Lifshitz, '**Classical Mechanics**', 3rd ed., Pergamon Press, 1976

11. L.Schulman '**Techniques and Applications of Path Integration**', Wiley, 1981

12. A.Shaikh, 'Canonical Transformations in Quantum Mechanics', I.C.T.P. preprint IC/87/39, 1987, to be published in J.Math.Phys., (1988).

13. A.Shaikh, 'The Trivialization of Constraints in the Quantum Theory', I.C.T.P. preprint IC/87/314, 1987, submitted for publication.